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Row and column-majorization on $\mathbf{M}_{n,m}$

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ABSTRACT

Let $\mathbf{M}_{n,m}$ be the set of all $n \times m$ matrices with entries in $\mathbb{R}$. For $A, B \in \mathbf{M}_{n,m}$, it is said that A is row majorized (respectively column-majorized) by B if every row (respectively column) of A is majorized by the corresponding row (respectively column) of B, i.e. for every i $(1 \leqslant i \leqslant n)$ there exists a doubly stochastic matrix D_i such that $A_{(i)} = B_{(i)}D_i$, where $A_{(i)}$ and $B_{(i)}$ are the ith rows of A and B respectively. In this paper the relations row and column-majorization on $\mathbf{M}_{n,m}$ are studied and also all linear operators $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserving (or strongly preserving) row or column-majorization will be characterized.

1. Introduction

An square matrix is called row stochastic if it is (componentwise) nonnegative and each row sum is 1. If A and A^t are both row stochastic then A is called doubly stochastic. The concept of vector majorization is a much studied concept in linear algebra and its applications. For $x, y \in \mathbb{R}^n$ it is said that x is majorized by y (denoted by $x \prec_{ls} y$) if there is a doubly stochastic matrix D satisfying $x = Dy$. Marshall et al. book [10] and Hardy et al. text [6] are the standard general references for majorization and inequalities. To find a sampling of the diverse areas in which majorization has been used see [2,3]. Several generalizations of this concept have also been introduced for matrices. Let $A, B \in \mathbf{M}_{n,m}$. It is said that A is (left) multivariate majorized by B (denoted by $A \prec_{ls} B$) if $A = DB$ for some $n \times n$ doubly stochastic matrix D, see [5,7]. It is said that A is ls-row majorized by B (denoted by $A \prec_{ls}^{row} B$) if every

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row of A is vector majorized by the corresponding row of B, see [4]. A linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ is said to be a linear preserver (respectively strong linear preserver) of a given relation $\sim$ on $\mathbf{M}_{n,m}$ if $X \sim Y$ implies that $TX \sim TY$ (respectively $X \sim Y$ if and only if $TX \sim TY$). The linear preservers and strong linear preservers of multivariate majorization are fully characterized in [5,7] as follows. The notation is defined right below.

Proposition 1.1 [5, Theorem 2]. *Let T be a linear operator on $\mathbf{M}_{n,m}$. Then T preserves multivariate majorization if and only if one of the following holds:*

(a) *There exist $A_1, \ldots, A_m \in \mathbf{M}_{n,m}$ such that $TX = \sum_{j=1}^{m} (\mathrm{tr}\, x_j) A_j$, where $x_j \in \mathbb{R}^n$ is the jth column of X.*
(b) *There exist $R, S \in \mathbf{M}_m$ and $P \in \mathcal{P}_n$ such that $TX = PXR + \mathbf{J}XS$.*

Proposition 1.2 [7, Theorem 2.4]. *Let T be a linear operator on $\mathbf{M}_{n,m}$. Then T strongly preserves multivariate majorization if and only if there exist $R, S \in \mathbf{M}_m$ and $P \in \mathcal{P}_n$ such that $R(R + nS)$ is invertible and $TX = PXR + \mathbf{J}XS$.*

Some of our notations are explained next.
$\mathbb{R}^n$: the set of all $n \times 1$ real column vectors.
$\mathbb{R}_n$: the set of all $1 \times n$ real row vectors.
$\mathcal{P}_n$: the set of all $n \times n$ permutation matrices.
e_i: the real column vector with 1 as ith component and 0 elsewhere.
$\mathrm{tr}\, x$: the summation of all components of a vector x in $\mathbb{R}^n$ or $\mathbb{R}_n$.
$\mathbf{M}_n$: the set of all $n \times n$ real matrices.
$\mathcal{R}_n$: the set of all $n \times n$ row stochastic matrices.
$\mathcal{S}_n$: the set of all $n \times n$ doubly stochastic matrices.
$\mathbf{J}$: the square matrix all of whose entries are 1 (the size of $\mathbf{J}$ is understood from the content).
$X = [x_1/x_2/\ldots/x_n]$: an $n \times m$ matrix whose rows are $x_1, x_2, \ldots, x_n \in \mathbb{R}_m$.
$X = [x_1 \mid x_2 \mid \ldots \mid x_m]$: an $n \times m$ matrix with $x_j \in \mathbb{R}^n$ as the jth column of X.
$\mathbb{N}_k$: the set $\{1, \ldots, k\}$.
$A_{(i)}$: the ith row of a matrix $A \in \mathbf{M}_{n,m}$.
$\overline{X}$: the vector $[\overline{x_1}/\ldots/\overline{x_n}] \in \mathbb{R}^n$ with components $\overline{x_i} = m^{-1}(x_{i1} + x_{i2} + \cdots + x_{im})$, for $i \in \mathbb{N}_n$.
$\mathbf{M}_n(\mathbb{R}_m)$: the set of all $n \times n$ matrices with entries in $\mathbb{R}_m$.
x^t: is the transpose of a vector x.
All kind of majorization concerning the paper are put in the following definition:

Definition 1.3. Let $A, B \in \mathbf{M}_{n,m}$.

(i) ls-majorization: $A \prec_{ls} B$ if $A = DB$ for some $D \in \mathcal{S}_n$.
(ii) rs-majorization: $A \prec_{rs} B$ if $A = BD$ for some $D \in \mathcal{S}_m$.
(iii) ls-column majorization: $A \prec_{ls}^{column} B$ if every column of A is ls-majorized by the corresponding column of B, i.e. $Ae_i \prec_{ls} Be_i$, for all $i \in \mathbb{N}_m$.
(iv) rs-row majorization: $A \prec_{rs}^{row} B$ if every row of A is rs-majorized by the corresponding row of B, i.e. $e_i^t A \prec_{rs} e_i^t B$, for all $i \in \mathbb{N}_n$.
(v) lw-majorization: $A \prec_{lw} B$ if $A = RB$ for some $R \in \mathcal{R}_n$.
(vi) rw-majorization: $A \prec_{rw} B$ if $A = BR$ for some $R \in \mathcal{R}_m$.
(vii) lw-column majorization: $A \prec_{lw}^{column} B$ if every column of A is lw-majorized by the corresponding column of B, i.e. $Ae_i \prec_{lw} Be_i$, for all $i \in \mathbb{N}_m$.
(viii) rw-row majorization: $A \prec_{rw}^{row} B$ if every row of A is rw-majorized by the corresponding row of B, i.e. $e_i^t A \prec_{rw} e_i^t B$, for all $i \in \mathbb{N}_n$.

Note that when A and B have a single row the notion of rs-majorization coincides with classical majorization. We have the following relations between the mentioned majorizations in the above definition. These properties can be shown directly from the definitions:

$A \prec_{ls} B \Longrightarrow A \prec_{lw} B$
$A \prec_{rs} B \Longrightarrow A \prec_{rw} B$
$A \prec_{ls}^{column} B \Longrightarrow A \prec_{lw}^{column} B$
$A \prec_{rs}^{row} B \Longrightarrow A \prec_{rw}^{row} B$
$A \prec_{ls} B \Longleftrightarrow A^t \prec_{rs} B^t$
$A \prec_{ls}^{column} B \Longleftrightarrow A^t \prec_{rs}^{row} B^t$

The linear preservers of some of the mentioned majorizations in Definition 1.3 have been found in [5,7–9]. The aim of this paper is to find the linear preservers of the remaining majorizations in Definition 1.3.

This paper is organized as follows: In Section 2 the linear preservers and strong linear preservers of *rs*-row (and *ls*-column) majorization on $\mathbf{M}_{n,m}$ will be characterized. In Section 3 the concept of *rw*-row (and *lw*-column) majorization on $\mathbf{M}_{n,m}$ are studied and their linear preservers and strong linear preservers will be characterized.

2. Linear preservers of *rs*-row (*ls*-column) majorization on $\mathbf{M}_{n,m}$

In this section we characterize linear operators $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ which preserve or strongly preserve *rs*-row (*ls*-column) majorization. First we need some known facts and lemmas. The following propositions have been proved in [1,5].

Proposition 2.1. *A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ preserves $\prec_{ls}$ if and only if one of the following holds:*

(i) $Tx = (\operatorname{tr} x)a$ for some $a \in \mathbb{R}^n$.
(ii) $Tx = \alpha Px + \beta \mathbf{J}x$ for some $\alpha, \beta \in \mathbb{R}$ and $P \in \mathcal{P}_n$.

Corollary 2.2. *A linear operator $T : \mathbb{R}_n \to \mathbb{R}_n$ preserves $\prec_{rs}$ if and only if one of the following holds:*

(i) $Tx = (\operatorname{tr} x)a$ for some $a \in \mathbb{R}_n$.
(ii) $Tx = \alpha xP + \beta x\mathbf{J}$ for some $\alpha, \beta \in \mathbb{R}$ and $P \in \mathcal{P}_n$.

Proposition 2.3 ([5, Lemma 3])**.** *Let $P \in \mathcal{P}_n$, $\alpha_1, \beta_1 \in \mathbb{R}$ with $\alpha_1 \neq 0$, and $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ satisfy $T_1x = \alpha_1 Px + \beta_1 \mathbf{J}x$. Suppose $T_2 : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_{ls}$ such that*

$$\gamma_1 T_1 Qy + \gamma_2 T_2 Qy \prec_{ls} \gamma_1 T_1 y + \gamma_2 T_2 y,$$

for all $\gamma_1, \gamma_2 \in \mathbb{R}$, $y \in \mathbb{R}^n$, $Q \in \mathcal{P}_n$. Then there exist $\alpha_2, \beta_2 \in \mathbb{R}$ such that $T_2 x = \alpha_2 Px + \beta_2 \mathbf{J}x$.

Corollary 2.4. *Let $P \in \mathcal{P}_n$, $\alpha_1, \beta_1 \in \mathbb{R}$ with $\alpha_1 \neq 0$, and $T_1 : \mathbb{R}_n \to \mathbb{R}_n$ satisfy $T_1 x = \alpha_1 xP + \beta_1 x\mathbf{J}$. Suppose $T_2 : \mathbb{R}_n \to \mathbb{R}_n$ is a linear preserver of $\prec_{rs}$ such that*

$$\gamma_1 T_1 yQ + \gamma_2 T_2 yQ \prec_{rs} \gamma_1 T_1 y + \gamma_2 T_2 y,$$

for all $\gamma_1, \gamma_2 \in \mathbb{R}$, $y \in \mathbb{R}_n$, $Q \in \mathcal{P}_n$. Then there exist $\alpha_2, \beta_2 \in \mathbb{R}$ such that $T_2 x = \alpha_2 xP + \beta_2 x\mathbf{J}$.

Proof. For every i $(1 \leqslant i \leqslant 2)$ define $\tau_i : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau_i x = [T_i x^t]^t$ for all $x \in \mathbb{R}_n$ and observe that Proposition 2.3 is applicable to τ_1 and τ_2. $\square$

Lemma 2.5. *Let $a \in \mathbb{R}_n$. The linear operator $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ defined by $TX = [aX/\ldots/aX]$, preserves $\prec_{rs}^{row}$ if and only if $a \in \cup_{i=1}^{n} \operatorname{Span}\{e_i^t\}$.*

Proof. If $a \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$, it is easy to show that T preserves $\prec_{rs}^{row}$. Conversely, let T preserve $\prec_{rs}^{row}$. Assume if possible $a = (a_1, \ldots, a_n) \notin \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$. Then there exist $i, j \in \mathbb{N}_n$ such that $a_i, a_j \neq 0$. Without loss of generality assume that $a_1, a_2 \neq 0$. Put

$$X := \begin{pmatrix} -a_2 & a_2 \\ -a_1 & a_1 \end{pmatrix} \oplus 0, \quad Y := \begin{pmatrix} a_2 & -a_2 \\ -a_1 & a_1 \end{pmatrix} \oplus 0 \in \mathbf{M}_{n,m}.$$

It is clear that $X \prec_{rs}^{row} Y$, so $aX \prec_{rs} aY$. But $aY = 0$ and $aX \neq 0$ which is a contradiction. $\square$

For every $i, j \in \mathbb{N}_n$, consider the embedding $E^j : \mathbb{R}_m \to \mathbf{M}_{n,m}$ by $E^j(x) = e_j x$ and projection $E_i : \mathbf{M}_{n,m} \to \mathbb{R}_m$ by $E_i(A) = e_i^t A$. It is easy to show that for every linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$,

$$TX = \left[\sum_{j=1}^{n} T_1^j x_j \Big/ \ldots \Big/ \sum_{j=1}^{n} T_n^j x_j \right],$$

where $T_i^j = E_i \circ T \circ E^j$ and $X = [x_1/\ldots/x_n]$. If T preserves $\prec_{rs}^{row}$, it is clear that $T_i^j : \mathbb{R}_m \to \mathbb{R}_m$ preserves $\prec_{rs}$. The following theorem characterizes the linear operators preserving $\prec_{rs}^{row}$ on $\mathbf{M}_{n,m}$.

Theorem 2.6. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T preserves $\prec_{rs}^{row}$ if and only if there exist $A_1, \ldots, A_n \in \boldsymbol{M}_{n,m}$, $b_1, \ldots, b_n \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$, $P_1, \ldots, P_n \in \mathcal{P}_m$ and $S \in \boldsymbol{M}_n$ such that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $e_i^t A_1 = \cdots = e_i^t A_n = 0$ and for all $X = [x_1/\ldots/x_n] \in \boldsymbol{M}_{n,m}$,*

$$TX = \sum_{j=1}^{n} (\mathrm{tr} x_j) A_j + [b_1 X P_1/\ldots/b_n X P_n] + SX\boldsymbol{J}. \tag{2.1}$$

Proof. First assume that T preserves $\prec_{rs}^{row}$. For every $i, j \in \mathbb{N}_n$, $T_i^j : \mathbb{R}_m \to \mathbb{R}_m$ preserves $\prec_{rs}$ as observed above. Then, each T_i^j is of the form (i) or (ii) in Corollary 2.2. Put

$$\mathbf{I} = \{k \in \mathbb{N}_n : \exists l \in \mathbb{N}_n, \alpha_k^l \neq 0, \beta_k^l \in \mathbb{R}, P_k \in \mathcal{P}_m, \; T_k^l x = \alpha_k^l x P_k + \beta_k^l x \mathbf{J}\}.$$

We show that if $k \in \mathbf{I}$, then T_k^j is of form (ii) with same $P_k \in \mathcal{P}_m$, for every $j \in \mathbb{N}_n$. Suppose $k \in \mathbf{I}$, then there exists $l \in \mathbb{N}_n$, $\alpha_k^l \neq 0$, $\beta_k^l \in \mathbb{R}$, $P_k \in \mathcal{P}_m$ such that $T_k^l x = \alpha_k^l x P_k + \beta_k^l x \mathbf{J}$. For every $x \in \mathbb{R}_m$, $\gamma_1, \gamma_2 \in \mathbb{R}$, set $X = \gamma_1 e_l x + \gamma_2 e_j x \in \mathbf{M}_{n,m}$. It is clear that $XQ \prec_{rs}^{row} X$, and hence $TXQ \prec_{rs}^{row} TX$, for all $Q \in \mathcal{P}_m$. This implies that

$$\gamma_1 T_k^l x Q + \gamma_2 T_k^j x Q \prec_{rs} \gamma_1 T_k^l x + \gamma_2 T_k^j x,$$

for all $\gamma_1, \gamma_2 \in \mathbb{R}$, $x \in \mathbb{R}_m$, $Q \in \mathcal{P}_m$. Then there exist $\alpha_k^j, \beta_k^j \in \mathbb{R}$ such that $T_k^j x = \alpha_k^j P_k x + \beta_k^j \mathbf{J} x$, by Corollary 2.4. Set $b_k := (\alpha_k^1, \ldots, \alpha_k^n)$, $S_{(k)} := (\beta_k^1, \ldots, \beta_k^n) \in \mathbb{R}_n$ if $k \in \mathbf{I}$ and $b_k = S_{(k)} := 0 \in \mathbb{R}_n$ if $k \notin \mathbf{I}$. Define $S := [S_{(1)}/\ldots/S_{(n)}] \in \mathbf{M}_{n,m}$.

If $k \notin \mathbf{I}$, then T_k^j is of form (i) for every $j \in \mathbb{N}_n$ and hence $T_k^j x = (\mathrm{tr} x) a_k^j$, for some $a_k^j \in \mathbb{R}_m$. For $k \in \mathbf{I}$, put $a_k^j = 0$ and define $A_j := [a_1^j/\ldots/a_n^j] \in \mathbf{M}_{n,m}$.

It is clear that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $e_i^t A_1 = \cdots = e_i^t A_n = 0$ and by a straightforward calculation one may show that for any $X = [x_1/\ldots/x_n] \in \mathbf{M}_{n,m}$,

$$TX = \sum_{j=1}^{n} (\mathrm{tr} x_j) A_j + [b_1 X P_1/\ldots/b_n X P_n] + SX\mathbf{J}.$$

If $b_j \notin \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$ for some $j \in \mathbb{N}_n$, then Lemma 2.5 implies that T is not a linear preserver of $\prec_{rs}^{row}$ which is a contradiction. Therefore $b_1, \ldots, b_n \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$, as desired. Conversely, let the condition (2.1) hold. Suppose $X = [x_1/\ldots/x_n]$, $Y = [y_1/\ldots/y_n] \in \mathbf{M}_{n,m}$ and $X \prec_{rs}^{row} Y$. Since for every $i \in \mathbb{N}_n$, $b_i = 0$ or $e_i^tA_1 = \cdots = e_i^tA_n = 0$, it is easy to see that $e_i^tTX \prec_{rs} e_i^tTY$ and hence $TX \prec_{rs}^{row} TY$. □

The following theorem gives the structure of strong linear preservers of $\prec_{rs}^{row}$ on $\mathbf{M}_{n,m}$.

Theorem 2.7. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Then T strongly preserves $\prec_{rs}^{row}$ if and only if there exist $P_1, \ldots, P_n \in \mathcal{P}_m$, $b_1, \ldots, b_n \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$, and $S \in \mathbf{M}_n$ such that $B(B + mS)$ is invertible and*

$$TX = [b_1XP_1/\ldots/b_nXP_n] + SX\boldsymbol{J}, \tag{2.2}$$

where $B = [b_1/\ldots/b_n]$.

Proof. The fact that the condition (2.2) is sufficient for T to be a strong linear preserver of $\prec_{rs}^{row}$ is easy to prove. So, we prove the necessity of the condition. Assume that T is a strong linear preserver of $\prec_{rs}^{row}$. It is easy to see that T is invertible. In view of Theorem 2.6, there exist $A_1, \ldots, A_n \in \mathbf{M}_{n,m}$, $b_1, \ldots, b_n \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$, $P_1, \ldots, P_n \in \mathcal{P}_m$ and $S \in \mathbf{M}_n$ such that $TX = \sum_{j=1}^{n}(\mathrm{tr}x_j)A_j + [b_1XP_1/\ldots/b_nXP_n] + SX\mathbf{J}$ and for every $i \in \mathbb{N}_n$, $b_i = 0$ or $e_i^tA_1 = \cdots = e_i^tA_n = 0$. We show that for every $j \in \mathbb{N}_n$, $A_j = 0$. Assume if possible there exists $j \in \mathbb{N}_n$, such that $A_j \neq 0$. Without loss of generality suppose that $e_1^tA_j \neq 0$, then $b_1 = 0$. Choose a nonzero $s \in (\mathrm{Span}\{b_2, \ldots, b_n\})^{\perp}$ and put $X := [s^t \mid -s^t \mid 0 \mid \ldots \mid 0]$. Then X is nonzero and $TX = 0$ that is a contradiction. Therefore $A_j = 0$, for every $j \in \mathbb{N}_n$.

Now, we show that B is invertible. Assume if possible B is not invertible, then there exists a nonzero $s \in (\mathrm{Span}\{b_1, \ldots, b_n\})^{\perp}$. Put $X := [s^t \mid -s^t \mid 0 \mid \ldots \mid 0]$ and conclude that X is nonzero and $TX = 0$ that is a contradiction. Therefore B is invertible.

Finally, we show that $B + mS$ is invertible. Assume if possible $B + mS$ is not invertible. Choose a nonzero $x \in \mathbb{R}^n$ such that $(B + mS)x = 0$ and put $X := [x \mid \ldots \mid x]$. Then X is nonzero and

$$TX = [b_1XP_1/\ldots/b_nXP_n] + SXJ = [(B + mS)x/\ldots/(B + mS)x] = 0,$$

that is a contradiction. Therefore $B + mS$ is invertible and the proof is complete. □

Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Define $\tau : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ by $\tau X = (TX^t)^t$. It is easy to see that T is a (strong) linear preserver of $\prec_{rs}^{row}$ if and only if τ is a (strong) linear preserver of $\prec_{ls}^{column}$. Combining this fact and previous theorems, we have the following theorems:

Theorem 2.8. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Then T preserves $\prec_{ls}^{column}$ if and only if there exist $A_1, \ldots, A_m \in \mathbf{M}_{n,m}$, $b_1, \ldots, b_m \in \cup_{i=1}^{m}\mathrm{Span}\{e_i\}$, $P_1, \ldots, P_m \in \mathcal{P}_n$ and $S \in \mathbf{M}_m$ such that for every $i \in \mathbb{N}_m$, $b_i = 0$ or $A_1e_i = \cdots = A_me_i = 0$ and for all $X = [x_1 \mid \ldots \mid x_m] \in \mathbf{M}_{n,m}$,*

$$TX = \sum_{j=1}^{m}(\mathrm{tr}x_j)A_j + [P_1Xb_1 \mid \ldots \mid P_mXb_m] + \boldsymbol{J}XS.$$

Theorem 2.9. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Then T strongly preserves $\prec_{ls}^{column}$ if and only if there exist $P_1, \ldots, P_m \in \mathcal{P}_n$, $b_1, \ldots, b_m \in \cup_{i=1}^{m}\mathrm{Span}\{e_i\}$, and $S \in \mathbf{M}_m$ such that $B(B + mS)$ is invertible and*

$$TX = [P_1Xb_1 \mid \ldots \mid P_mXb_m] + \boldsymbol{J}XS,$$

where $B = [b_1 \mid \ldots \mid b_m]$.

3. Linear preservers of $\prec_{rw}^{row}$ (or $\prec_{lw}^{column}$) on $\mathbf{M}_{n,m}$

In this section we characterize linear operators $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ which preserve or strongly preserve $\prec_{rw}^{row}$ (or $\prec_{lw}^{column}$). Through this section we assume that $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proposition 3.1 ([9, Theorem 3]). *Let $\alpha, \beta \in \mathbb{R}$ with $0 \leqslant min\{\alpha, \beta\} < max\{\alpha, \beta\} = 1$, and $T_1 : \mathbb{R}_m \to \mathbb{R}_m$ satisfy $T_1x = \lambda x(\alpha I + \beta Q)$. Suppose $T_2 : \mathbb{R}_m \to \mathbb{R}_m$ is a linear preserver of $\prec_{rw}$ such that*

$$T_1xR + T_2yR \prec_{rw} T_1x + T_2y,$$

for all $x, y \in \mathbb{R}_m$, $R \in \mathcal{R}_m$. Then there exists $\mu \in \mathbb{R}$ such that $T_2x = \mu x(\alpha I + \beta Q)$.

Theorem 3.2 ([9, Theorem 2.1]). *If a linear operator $T : \mathbb{R}_m \to \mathbb{R}_m$ preserves $\prec_{rw}$, then one of the following condition holds:*

(i) $rank(T) \leqslant 1$ and, in fact, there exists $a \in \mathbb{R}_m$ such that $TX = maX$ for all $X \in \mathbb{R}_m$.
(ii) $rank(T) = m = 2$ and $TX = lX(cI + dQ)$ for all $X \in \mathbb{R}_m$, where $0 \neq l \in \mathbb{R}$ and $0 \leqslant min\{c, d\} < max\{c, d\} = 1$.
(iii) $rank(T) = m \geqslant 3$ and $TX = lXP$ for all $X \in \mathbb{R}_m$, where $0 \neq l \in \mathbb{R}$ and $P \in \mathcal{P}_m$.

Lemma 3.3. *Let $a \in \mathbb{R}_n$. The linear operator $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ defined by $TX = [aX/\dots/aX]$, preserves rw-row majorization if and only if $a \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$.*

Proof. If $a \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$, it is easy to show that T preserves rw-row majorization. Conversely, let T preserve rw-row majorization. Assume if possible $a = (a_1, \dots, a_n) \notin \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$. Then there exist $i, j \in \mathbb{N}_n$ such that $a_i, a_j \neq 0$. Without loss of generality assume that $a_1, a_2 \neq 0$. Consider $X = \begin{pmatrix} -a_2 & a_2 \\ -a_1 & a_1 \end{pmatrix} \oplus 0$, $Y = \begin{pmatrix} a_2 & -a_2 \\ -a_1 & a_1 \end{pmatrix} \oplus 0$. It is clear that $X \prec_{rw}^{row} Y$, so $aX \prec_{rw} aY$. But $aY = 0$ and $aX \neq 0$, which is a contradiction. $\square$

Theorem 3.4. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T preserves rw-row majorization if and only if one of the following conditions holds:*

(a) $m = 2$ and there exist $\mathcal{A} \in \boldsymbol{M}_n(\mathbb{R}_m)$, $b_1, \dots, b_n \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$, $\alpha_i, \beta_i \in \mathbb{R}$ with $0 \leqslant min\{\alpha_i, \beta_i\} < max\{\alpha_i, \beta_i\} = 1$ such that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $\mathcal{A}_{(i)} = 0$ and

$$TX = m\mathcal{A}\overline{X} + [b_1X(\alpha_1I + \beta_1Q)/\dots/b_nX(\alpha_nI + \beta_nQ)].$$

(b) $m \geqslant 3$ and there exist $\mathcal{A} \in \boldsymbol{M}_n(\mathbb{R}_m)$, $b_1, \dots, b_n \in \cup_{i=1}^{n}\mathrm{Span}\{e_i^t\}$ and $P_1, \dots, P_n \in \mathcal{P}_m$ such that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $\mathcal{A}_{(i)} = 0$ and

$$TX = m\mathcal{A}\overline{X} + [b_1XP_1/\dots/b_nXP_n].$$

Proof. The fact that the condition (a) or (b) is sufficient for T to be a strong linear preserver of $\prec_{rw}^{row}$ is easy to prove. So, we prove the necessity of the conditions. Now, assume that T preserves $\prec_{rw}^{row}$. Let $m = 2$. For every $i, j \in \mathbb{N}_n$, $T_i^j : \mathbb{R}_m \to \mathbb{R}_m$ preserves $\prec_{rw}$. Then, each T_i^j is of the form (i) or (ii) in Theorem 3.2. Put

$$\mathbf{I} = \{k \in \mathbb{N}_n : \exists l \in \mathbb{N}_n, \lambda_k^l \neq 0, \alpha_k, \beta_k \in \mathbb{R}, T_k^l x = \lambda_k^l x(\alpha_k^l I + \beta_k^l Q)\},$$

where $0 \leqslant min\{\alpha_k, \beta_k\} < max\{\alpha_k, \beta_k\} = 1$. We show that if $k \in \mathbf{I}$, then T_k^j is of form (ii) of Theorem 3.2 for every $j \in \mathbb{N}_n$. Suppose $k \in \mathbf{I}$, then there exist $l \in \mathbb{N}_n$, $\lambda_k^l \neq 0 \in \mathbb{R}$, $\alpha_k, \beta_k \in \mathbb{R}$ such that

$0 \leqslant min\{\alpha_k, \beta_k\} < max\{\alpha_k, \beta_k\} = 1$ and $T_k^l x = \lambda_k^l x(\alpha_k I + \beta_k Q)$. Set $X = e_l x + e_j y$. It is clear that $XR \prec_{rw}^{row} X$ and hence $TXR \prec_{rw}^{row} TX$ for all $R \in \mathcal{R}_m$. This implies that

$$T_k^l xR + T_k^j yR \prec_{rw} T_k^l x + T_k^j y,$$

for all $x, y \in \mathbb{R}_m$. So by Proposition 3.1, there exists $\lambda_k^j \in \mathbb{R}$ such that $T_k^j x = \lambda_k^j x(\alpha_k I + \beta_k Q)$. Set $b_k := (\lambda_k^1, \ldots, \lambda_k^n)$ if $k \in \mathbf{I}$, and $b_k = 0$ if $k \notin \mathbf{I}$.
If $k \notin \mathbf{I}$, then T_k^j is of form (i) of Theorem 3.2 for every $j \in \mathbb{N}_n$ and hence $T_k^j x = ma_k^j x$ for some $a_k^j \in \mathbb{R}_m$. If $k \in \mathbf{I}$, put $a_k^j = 0$ for every $j \in \mathbb{N}_n$. For $k \in \mathbb{N}_n$ define $\mathcal{A}_{(k)} = [a_k^1 \ldots a_k^n]$.
It is clear that for every $i \in \mathbb{N}_n$, $b_i = 0$ or $\mathcal{A}_{(i)} = 0$. Define $\mathcal{A} = [\mathcal{A}_{(1)}/\ldots/\mathcal{A}_{(n)}]$, then $TX = [\sum_{j=1}^n T_i^j x_j/\ldots/\sum_{j=1}^n T_i^j x_j] = m\mathcal{A}\overline{X} + [b_1 X(\alpha_1 I + \beta_1 Q)/\ldots/b_n X(\alpha_n I + \beta_n Q)]$. For $m \geqslant 3$, with an argument same as above it is showed that $TX = m\mathcal{A}\overline{X} + [b_1 XP_1/\ldots/b_n XP_n]$. $\square$

Theorem 3.5. *Let $m \geqslant 3$. $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ is a strong linear preserver of $\prec_{rw}^{row}$ if and only if there exist $b_1, \ldots, b_n \in \cup_{i=1}^n \mathrm{Span}\{e_i^t\}$, $P_1, \ldots, P_n \in \mathcal{P}_m$ such that $B := [b_1/\ldots/b_n]$ is invertible and*

$$TX = [b_1 XP_1/\ldots/b_n XP_n].$$

Proof. Assume that there exist $k \notin I$. Without loss of generality assume that $1 \notin I$, so $b_1 = 0$. Set $V := \mathrm{span}\{b_2, \ldots, b_n\}$, so $dimV \leqslant n-1$. Therefore $dimV^\perp \geqslant 1$ and there exists $0 \neq s \in V^\perp$. Set $X := [s \mid -s \mid 0 \mid \ldots \mid 0]$. Therefore X is nonzero and $b_i X = 0$, so $TX = 0$, that is a contradiction. So for every $k \in \mathbb{N}_n$, $k \in I$. Therefore $TX = [b_1 XP_1/\ldots/b_n XP_n]$.
Now we show that B is invertible. Let if possible B is not invertible, and set $V := \mathrm{span}\{b_1, \ldots, b_n\}$. So $dimV \leqslant n-1$. Therefore $dimV^\perp \geqslant 1$ and there exists $0 \neq s \in V^\perp$. Set $X := [s \mid -s \mid 0 \mid \ldots \mid 0]$. Therefore X is nonzero and $b_i X = 0$, then $TX = 0$ which is a contradiction. $\square$

Corollary 3.6. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a strong linear preserver of $\prec_{rw}^{row}$. Then T strongly preserves $\prec_{rs}^{row}$.*

Now we characterize all linear operators $T : M_{n,m} \to M_{n,m}$ that preserve or strongly preserve lw-column majorization.

Theorem 3.7 ([8, Theorem 2.3])**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then T preserves $\prec_{lw}$ if and only if $TX = (aI + bP)X$ for all $x \in \mathbb{R}^n$, where $P \neq I$ is an $n \times n$ permutation matrix, and $a, b \in \mathbb{R}$ are such that $ab \leqslant 0$ and, if $n \neq 2$, $ab = 0$. In case $n \neq 2$, then $aI + bP = cQ$ for some $c \in \mathbb{R}$ and, hence, $TX = QXK$ for some $K \in M_m$.*

Lemma 3.8. *Let $a \in \mathbb{R}^m$. The linear operator $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ defined by $TX = [Xa \mid \ldots \mid Xa]$, preserves lw-column-majorization if and only if $a \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$.*

Theorem 3.9. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear operator. Then T preserves lw-column majorization if and only if one of the following condition holds:*

(a) $n = 2$ and there exist $b_1, \ldots, b_m \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$, $Q \neq I \in \mathcal{P}_n$, $\alpha_i, \beta_i \in \mathbb{R}$ such that $\alpha_i \beta_i \neq 0$, and

$$TX = [(\alpha_1 I + \beta_1 Q)Xb_1 \mid \ldots \mid (\alpha_n I + \beta_n Q)Xb_m].$$

(b) $n \neq 2$ and there exist $b_1, \ldots, b_m \in \cup_{i=1}^m \mathrm{Span}\{e_i\}$ and $P_1, \ldots, P_m \in \mathcal{P}_n$ such that

$$TX = [P_1 Xb_1 \mid \ldots \mid P_m Xb_m].$$

Theorem 3.10. *Let $T : M_{n,m} \to M_{n,m}$ be a linear operator. Then T strongly preserves lw-column majorization if and only if there exists $b_1, \ldots, b_m \in \cup_{i=1}^{m} \mathrm{Span}\{e_i\}, P_1, \ldots, P_m \in \mathcal{P}_n$ such that $B := [b_1 \mid \ldots \mid b_m]$ is invertible, $TX = [P_1Xb_1 \mid \ldots \mid P_mXb_m]$.*

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References

[1] T. Ando, Majorization doubly stochastic matrices, and comparison of eigenvalues, Linear Algebra Appl. 118 (1989) 163–248.
[2] B.C. Arnold, Majorization: here, there and everywhere, Statist. Sci. 22 (3) (2007) 407–413.
[3] M. Buliga, Four applications of majorization to convexity in the calculus of variations, Linear Algebra Appl. 429 (2008) 1528–1545.
[4] R.A. Brualdi, G. Dahl, Majorization classes of integral matrices, Linear Algebra Appl. 436 (2012) 802–813.
[5] C.K. Li, E. Poon, Linear operators preserving directional majorization, Linear Algebra Appl. 325 (2001) 15–21.
[6] G. Hardy, J.E. Littlewood, G. Polya, Inequalities, second ed., Cambridge University Press, Cambridge, 1952.
[7] A.M. Hasani, M. Radjabalipuor, The structure of linear operators strongly preserving majorizations of matrices, Electron. J. Linear Algebra 15 (2006) 260–268.
[8] A.M. Hasani, M. Radjabalipuor, Linear preserver of matrix majorization, Internat. J. Pure Appl. Math. 32 (2006) 475–482.
[9] A.M. Hasani, M. Radjabalipuor, On linear preserves of (right) matrix majorization, Linear Algebra Appl. 423 (2007) 255–261.
[10] A.W. Marshall, I. Olkin, B.C. Arnold, Inequalities: Theory of Majorizations and its Applications, second ed., Springer, New York, 2011.